Option Pricing Models

Evaluation and Implementation

Comparative Analysis of Black–Scholes, Binomial (CRR) and Trinomial (Boyle)

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Abstract

This document presents a comprehensive comparative study of three foundational option pricing frameworks: the analytical Black–Scholes model, the Cox–Ross–Rubinstein (CRR) binomial tree, and the Boyle trinomial tree. We implement a modern web application (Next.js + FastAPI) delivering real-time pricing, Greeks computation, and comprehensive diagnostics including convergence analysis, arbitrage checks, and put–call parity verification. While Black–Scholes serves as the gold standard for European options due to its closed-form solution, tree-based methods provide essential flexibility for American exercise features. We rigorously document numerical behavior, analyze discretization choices, and demonstrate convergence to the Black–Scholes benchmark under standard market parameterizations.

1 Introduction

Accurate option pricing is fundamental to modern financial markets, serving as the cornerstone for trading strategies, risk management frameworks, and derivative product design. The **Black–Scholes** (BS) model revolutionized quantitative finance by providing an elegant closed-form solution for European vanilla options under the assumption of geometric Brownian motion (GBM) with constant volatility and interest rates. However, practical market requirements—most notably **American exercise** features allowing early termination—necessitate numerical approximation techniques.

Discrete-time lattice methods, particularly the **Cox–Ross–Rubinstein (CRR) binomial** framework and the **Boyle trinomial** construction, approximate continuous-time dynamics through recombining tree structures. These methods enable backward induction algorithms that naturally accommodate early-exercise decisions at each node, making them indispensable tools for pricing American-style derivatives.

Document Structure

Section 1: Introduction
Section 2: Theoretical Framework — Mathematical foundations, probabilistic setup, and model derivations
Section 3: Implementation — Computational algorithms, web architecture, validation and Results
Section 5: Conclusion

1.1 Motivation and Objectives

This project bridges theoretical option pricing with practical implementation, pursuing three primary objectives:

1. **Theoretical Rigor**: Develop complete mathematical derivations from first principles, including risk-neutral valuation, martingale theory, and convergence proofs
2. **Computational Efficiency**: Implement optimized algorithms achieving $O(N)$ space complexity and practical runtime performance
3. **Practical Accessibility**: Deploy an interactive web platform enabling real-time pricing, sensitivity analysis, and educational exploration

1.2 Key Contributions

- Complete theoretical framework with rigorous proofs of convergence
- Efficient implementations of CRR binomial and Boyle trinomial algorithms
- Comprehensive validation suite (put-call parity, arbitrage bounds, Greeks)
- Interactive web application with real-time computation and visualization

2 Theoretical Framework

2.1 Probabilistic Workspace

We establish the mathematical foundations for option pricing within a rigorous probability-theoretic framework.

2.1.1 Filtered Probability Space

Consider Ω the set of all market states, equipped with:

- A probability measure $\mathbb{P}$ (real-world probabilities)
- A filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t\geq 0}$ (flow of information, $\mathcal{F}_t$ is information at t)
- The quadruple $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ forms our filtered probability space

Filtration Property: The filtration satisfies $\mathcal{F}_s \subseteq \mathcal{F}_t$ for $s \leq t$, ensuring information accumulates and is never lost. This monotonicity is crucial for defining adapted processes.

2.1.2 Market Structure and Fundamental Assumptions

Core Market Assumptions

1. No-Arbitrage Assumption (AOA)
There are no strategies yielding strictly positive wealth at time T from zero initial capital: $\nexists \phi$ such that $X_0^\phi = 0$ and $\mathbb{P}(X_T^\phi \geq 0) = 1$ with $\mathbb{P}(X_T^\phi > 0) > 0$.
2. Market Completeness
Every contingent claim (derivative payoff) is replicable by a self-financing trading strategy.
3. Frictionless Markets
No transaction costs, taxes, or bid-ask spreads. Continuous trading and infinite divisibility are assumed.

The market comprises two primitive assets:

1. **Risk-free asset (B_t)**: Evolves deterministically as $B_t = B_0 e^{rt}$, with risk-free rate $r \geq 0$.
2. **Risky asset (S_t)**: Stock price follows a stochastic process adapted to $\mathcal{F}_t$.

Fundamental Theorem of Asset Pricing (FTAP)

Under the no-arbitrage assumption:

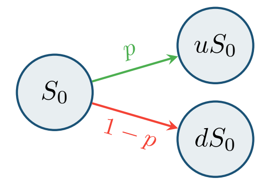
1. There exists a risk-neutral probability measure $\mathbb{Q} \sim \mathbb{P}$ (equivalent)
2. All discounted asset prices are $\mathbb{Q}$-martingales: $\frac{S_t}{B_t} = \mathbb{E}^{\mathbb{Q}}\left[\frac{S_T}{B_T}\middle|\mathcal{F}_t\right]$
3. If the market is complete, $\mathbb{Q}$ is unique
4. Any derivative with payoff $\Phi(S_T)$ has price $V_t = B_t \cdot \mathbb{E}^{\mathbb{Q}}\left[\frac{\Phi(S_T)}{B_T}\middle|\mathcal{F}_t\right]$

2.2 One-Period Binomial Model

The binomial model discretizes time and restricts price movements to two possibilities per period, yielding tractable closed-form solutions while preserving essential market properties.

2.2.1 Model Setup

Consider two time points: $t_0 = 0$ (present) and $t_1 = T$ (maturity). The risky asset price S_0 at time 0 evolves to one of two values at time T:



Formal Definition:

- Sample space: $\Omega = \{\omega_u, \omega_d\}$ (up and down states)
- Filtration: $\mathcal{F}_0 = \{\emptyset, \Omega\}$ (trivial), $\mathcal{F}_1 = \sigma(S_1) = 2^\Omega$ (full information)
- Physical probabilities: $\mathbb{P}(\omega_u) = p$, $\mathbb{P}(\omega_d) = 1-p$ where $0 < p < 1$
- Price movements: $u > 1$ (up factor), $0 < d < 1$ (down factor)
- Risk-free return: $R = 1 + r = e^{rT}$ for continuous compounding

No-Arbitrage Constraint: To prevent arbitrage, we require $d < R < u$. If $R \leq d$, a strategy of shorting the stock and investing in bonds yields riskless profit. If $R \geq u$, borrowing to buy stock guarantees profit.

2.2.2 Replication Strategy

In a complete market, any derivative with payoff $C_1(\omega)$ at time T can be replicated by a portfolio strategy (ξ, Δ):

- ξ: Units of risk-free bond (initial capital: $x = \xi B_0$)
- Δ: Units of risky stock (delta hedge)

Portfolio value at maturity:

$$V_1(\omega) = \xi B_1 + \Delta S_1(\omega) = \xi R + \Delta S_0 Y_1(\omega) \tag{1}$$

where $Y_1 \in \{u, d\}$ is the return factor.
Replication equations:

$$\begin{cases} C_1^u = \xi R + \Delta S_0 u \\ C_1^d = \xi R + \Delta S_0 d \end{cases} \tag{2}$$

Solving this 2×2 linear system:

$$\Delta = \frac{C_1^u - C_1^d}{S_0(u-d)} \quad \text{(delta hedge ratio)} \tag{3}$$

$$\xi = \frac{1}{R}\left[C_1^u - \Delta S_0 u\right] = \frac{uC_1^d - dC_1^u}{R(u-d)} \tag{4}$$

Derivative Price: The initial capital required to replicate the derivative is:

$$C_0 = \xi + \Delta S_0 = \frac{1}{R}\left[qC_1^u + (1-q)C_1^d\right] \tag{5}$$

where the **risk-neutral probability** is:

$$q = \frac{R-d}{u-d}, \quad 1-q = \frac{u-R}{u-d} \tag{6}$$

2.2.3 Risk-Neutral Valuation

The risk-neutral measure $\mathbb{Q}$ transforms the pricing problem into an expectation under a probability measure where all assets earn the risk-free rate on average.

Theorem 1 (Risk-Neutral Pricing)**.** *Under $\mathbb{Q}$, the discounted stock price is a martingale:*

$$\frac{S_0}{B_0} = \mathbb{E}^{\mathbb{Q}}\left[\frac{S_1}{B_1}\right] \tag{7}$$

Any derivative with payoff C_1 has present value:

$$C_0 = \frac{1}{R}\mathbb{E}^{\mathbb{Q}}[C_1] = \frac{1}{R}\left[qC_1^u + (1-q)C_1^d\right] \tag{8}$$

Key Insight: The risk-neutral probability q is *not* the real-world probability p. It's determined entirely by market parameters (S_0, u, d, r) through the no-arbitrage condition. Investors' risk preferences (embedded in p) are irrelevant for pricing—a profound result known as **risk-neutral valuation**.

2.3 Multi-Period Binomial Model

We extend the one-period framework to N discrete time steps, creating a recombining tree structure that approximates continuous-time dynamics.

2.3.1 Temporal Discretization

Partition the time interval $[0, T]$ into N equal subintervals:

$$0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T, \quad \Delta t = \frac{T}{N} \tag{9}$$

At each time t_n, the stock price is:

$$S_{t_n} = S_0 \prod_{i=1}^{n} Y_i \tag{10}$$

where $Y_i \in \{u, d\}$ are i.i.d. return factors with $\mathbb{Q}(Y_i = u) = q$.

2.3.2 Tree Structure

The binomial tree has:

- **Nodes**: $(N+1)(N+2)/2$ total nodes
- **Terminal nodes**: $N+1$ possible values $S_T = S_0 u^j d^{N-j}$ for $j \in \{0, 1, \ldots, N\}$
- **Recombining property**: Up-then-down equals down-then-up ($S_0 ud = S_0 du$)

2.3.3 Backward Induction Pricing

Algorithm 1 Multi-Period Binomial Pricing

```
Step 1: Terminal Payoffs
for j = 0 to N do
    S_{N,j} = S_0 u^j d^{N-j}
    V_{N,j} = Φ(S_{N,j})    (e.g., (S_{N,j} − K)^+ for call)
end for
Step 2: Backward Recursion
for n = N − 1 down to 0 do
    for j = 0 to n do
        V_{n,j} = e^{−rΔt}[qV_{n+1,j+1} + (1 − q)V_{n+1,j}]
        For American: V_{n,j} = max(V_{n,j}, Intrinsic_{n,j})
    end for
end for
Result: C_0 = V_{0,0}
```

Closed-Form Expression: For European options:

$$C_0 = e^{-rT}\sum_{j=0}^{N}\binom{N}{j} q^j(1-q)^{N-j}\Phi(S_0 u^j d^{N-j}) \tag{11}$$

For a European call with strike K:

$$C_0 = e^{-rT}\sum_{j=j^*}^{N}\binom{N}{j} q^j(1-q)^{N-j}(S_0 u^j d^{N-j} - K) \tag{12}$$

where $j^* = \min\{j : S_0 u^j d^{N-j} > K\}$.

2.3.4 Cox–Ross–Rubinstein (CRR) Parameterization

The CRR model chooses specific values for u, d to ensure convergence to the continuous-time Black–Scholes model:

CRR Parameters

$$u = e^{\sigma\sqrt{\Delta t}} \tag{13}$$

$$d = e^{-\sigma\sqrt{\Delta t}} = \frac{1}{u} \tag{14}$$

$$R = e^{r\Delta t} \tag{15}$$

$$q = \frac{e^{r\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}} \tag{16}$$

where σ is the volatility of the underlying asset.

Key Properties:

1. $ud = 1$ (symmetry around S_0)
2. $\ln u = -\ln d = \sigma\sqrt{\Delta t}$
3. As $N \to \infty$, the CRR model converges to Black–Scholes (proven in Section 2.4.2)

2.4 Black–Scholes Model

2.4.1 Continuous-Time Framework

The Black–Scholes model assumes the stock price follows a geometric Brownian motion (GBM) under the physical measure:

$$dS_t = \mu S_t dt + \sigma S_t dB_t \tag{17}$$

where:

- μ: Expected return (drift)
- $\sigma > 0$: Volatility (constant)
- B_t: Standard Brownian motion under $\mathbb{P}$

Integrated form:

$$S_t = S_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t\right] \tag{18}$$

Black–Scholes Assumptions

1. Asset follows GBM with constant σ
2. Risk-free rate r is constant and known
3. No dividends
4. No transaction costs or taxes
5. Continuous trading possible
6. No arbitrage opportunities

2.4.2 Risk-Neutral Dynamics

Under the risk-neutral measure $\mathbb{Q}$ (obtained via Girsanov's theorem), the drift changes from μ to r:

$$dS_t = rS_t dt + \sigma S_t dW_t \tag{19}$$

where $W_t = B_t + \frac{\mu - r}{\sigma}t$ is a $\mathbb{Q}$-Brownian motion.

2.4.3 Black–Scholes Formula

Black–Scholes Pricing Formulas

For European options with strike K and maturity T:
Call Option:

$$C(S_0, T) = S_0 N(d_1) - Ke^{-rT}N(d_2) \tag{20}$$

Put Option:

$$P(S_0, T) = Ke^{-rT}N(-d_2) - S_0 N(-d_1) \tag{21}$$

where:

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \tag{22}$$

$$d_2 = d_1 - \sigma\sqrt{T} = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} \tag{23}$$

and $N(\cdot)$ is the standard normal CDF:
$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-z^2/2}dz$

Intuitive Interpretation:

- $N(d_1)$: Delta hedge ratio (probability of ITM under stock measure)
- $N(d_2)$: Risk-neutral probability of exercise
- $S_0 N(d_1)$: Expected present value of receiving stock if ITM
- $Ke^{-rT}N(d_2)$: Expected present value of paying strike if exercised

2.4.4 Convergence of Binomial to Black–Scholes

We now prove that the CRR binomial model converges to Black–Scholes as $N \to \infty$.

Theorem 2 (Binomial Convergence)**.** *Let C_N denote the CRR binomial price with N steps. Then:*

$$\lim_{N\to\infty} C_N = C_{BS} \tag{24}$$

where C_{BS} is the Black–Scholes price.

Proof Sketch. **Step 1: Terminal distribution.** With CRR parameterization:

$$S_T = S_0 \prod_{i=1}^{N} Y_i = S_0 e^{\sigma\sqrt{\Delta t}\sum_{i=1}^{N} Z_i} \tag{25}$$

where $Z_i \in \{-1, +1\}$ with $\mathbb{Q}(Z_i = 1) = q$.

Define $Z_N = \frac{1}{\sqrt{N}}\sum_{i=1}^{N} Z_i$. Under $\mathbb{Q}$, $Z_i \sim \text{Bernoulli}(q)$ rescaled to $\{-1, +1\}$.

Step 2: Moments analysis. Using Taylor expansion as $\Delta t \to 0$:

$$q = \frac{e^{r\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}} \tag{26}$$

$$= \frac{1}{2} + \frac{\sqrt{\Delta t}}{2\sigma}\left(r - \frac{\sigma^2}{2}\right) + O(\Delta t) \tag{27}$$

Mean and variance of Z_N:

$$\mathbb{E}^{\mathbb{Q}}[Z_N] = \frac{\sqrt{T}}{\sigma}\left(r - \frac{\sigma^2}{2}\right) + O(\Delta t) \tag{28}$$

$$\text{Var}^{\mathbb{Q}}(Z_N) = 1 - \frac{\Delta t}{\sigma^2}\left(r - \frac{\sigma^2}{2}\right)^2 + O(\Delta t^{3/2}) \tag{29}$$

Step 3: Central Limit Theorem. As $N \to \infty$:

$$Z_N \xrightarrow{d} \mathcal{N}\left(\frac{\sqrt{T}}{\sigma}\left(r - \frac{\sigma^2}{2}\right), 1\right) \tag{30}$$

Therefore:

$$\ln(S_T/S_0) = \sigma\sqrt{T}Z_N \xrightarrow{d} \mathcal{N}\left(\left(r - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right) \tag{31}$$

Step 4: Price convergence. Using dominated convergence:

$$C_N = e^{-rT}\mathbb{E}^{\mathbb{Q}}[(S_T - K)^+] \tag{32}$$

$$\to e^{-rT}\int_{-\infty}^{\infty}(S_0 e^x - K)^+ \frac{1}{\sqrt{2\pi\sigma^2 T}} e^{-\frac{(x - mT)^2}{2\sigma^2 T}} dx \tag{33}$$

where $m = r - \sigma^2/2$. Standard calculation yields the Black–Scholes formula. □

2.4.5 The Greeks: Sensitivity Analysis

Greeks measure how option prices respond to changes in market parameters, essential for risk management and hedging.

Delta (Δ) — Price Sensitivity

Definition: Rate of change with respect to underlying price

$$\Delta_{\text{Call}} = \frac{\partial C}{\partial S} = N(d_1) \tag{34}$$

$$\Delta_{\text{Put}} = \frac{\partial P}{\partial S} = N(d_1) - 1 = -N(-d_1) \tag{35}$$

Properties:

- Call: $\Delta \in (0, 1)$ — increases with moneyness
- Put: $\Delta \in (-1, 0)$ — becomes more negative as ITM
- ATM options: $\Delta \approx \pm 0.5$
- Approximates $\mathbb{Q}$(option expires ITM)

Gamma (Γ) — Convexity

Definition: Rate of change of delta (second derivative)

$$\Gamma = \frac{\partial^2 V}{\partial S^2} = \frac{N'(d_1)}{S\sigma\sqrt{T}} = \frac{\phi(d_1)}{S\sigma\sqrt{T}} \tag{36}$$

where $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ is the standard normal PDF.

Properties:

- Always positive for long options (calls and puts)
- Maximum at $S = K$ (ATM)
- Increases as expiration approaches
- Measures hedging error and portfolio rebalancing needs

Vega ($\mathcal{V}$) — Volatility Sensitivity

Definition: Sensitivity to implied volatility changes

$$\mathcal{V} = \frac{\partial V}{\partial \sigma} = S\sqrt{T}\,\phi(d_1) = S\sqrt{T}\,N'(d_1) \tag{37}$$

Properties:

- Positive for long options (benefit from vol increase)
- Maximum at $S = K$ (ATM)
- Proportional to $\sqrt{T}$ (longer maturity $\implies$ higher vega)
- Critical for volatility trading strategies

Theta (Θ) — Time Decay

Definition: Rate of value loss as time passes

$$\Theta_{\text{Call}} = -\frac{S\sigma\phi(d_1)}{2\sqrt{T}} - rKe^{-rT}N(d_2) \tag{38}$$

$$\Theta_{\text{Put}} = -\frac{S\sigma\phi(d_1)}{2\sqrt{T}} + rKe^{-rT}N(-d_2) \tag{39}$$

Properties:

- Typically negative for long options (time decay)
- Accelerates near expiration (especially ATM)
- Trade-off with gamma: high gamma $\implies$ high theta

Rho (ρ) — Interest Rate Sensitivity

Definition: Sensitivity to risk-free rate changes

$$\rho_{\text{Call}} = KTe^{-rT}N(d_2) \tag{40}$$

$$\rho_{\text{Put}} = -KTe^{-rT}N(-d_2) \tag{41}$$

Properties:

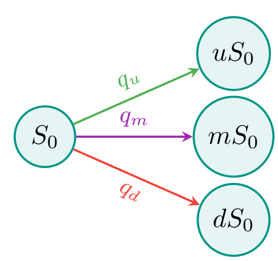
- Calls: positive (benefit from rate increases)
- Puts: negative (lose value when rates rise)
- More significant for long-dated options
- Generally least important Greek for equity options

2.5 Trinomial Model (Boyle)

The trinomial framework extends the binomial model by allowing three possible outcomes per period, providing enhanced flexibility and improved convergence properties.

2.5.1 One-Period Trinomial Structure

At each time step, the stock can move to three states:



Formal setup:

- State space: $\Omega = \{\omega_u, \omega_m, \omega_d\}$
- Price factors: $u > m > d$ with $d < 1 < u$
- Risk-neutral probabilities: $q_u + q_m + q_d = 1$
- No-arbitrage: $d < R < u$ and typically $m \le R$

Market Completeness Issue: With three states but only two assets (stock and bond), the market is incomplete—multiple risk-neutral measures exist. To restore completeness, we can either:

1. Add a third traded asset
2. Impose additional constraints (e.g., variance matching)

We adopt the second approach for practical implementation.

2.5.2 Boyle Parameterization

The Boyle (1988) scheme ensures recombination and convergence to Black–Scholes:

Boyle Parameters

$$u = e^{\sigma\sqrt{2\Delta t}} \tag{42}$$

$$d = e^{-\sigma\sqrt{2\Delta t}} = \frac{1}{u} \tag{43}$$

$$m = 1 \quad \text{(no change)} \tag{44}$$

$$R = e^{r\Delta t} \tag{45}$$

Risk-neutral probabilities (via variance matching):

$$q_u = \left(\frac{e^{r\Delta t/2} - e^{-\sigma\sqrt{\Delta t/2}}}{e^{\sigma\sqrt{\Delta t/2}} - e^{-\sigma\sqrt{\Delta t/2}}}\right)^2 \tag{46}$$

$$q_d = \left(\frac{e^{\sigma\sqrt{\Delta t/2}} - e^{r\Delta t/2}}{e^{\sigma\sqrt{\Delta t/2}} - e^{-\sigma\sqrt{\Delta t/2}}}\right)^2 \tag{47}$$

$$q_m = 1 - q_u - q_d \tag{48}$$

Key advantages:

1. **Recombining tree:** $umd = m$, reducing complexity from $O(3^N)$ to $O(N^2)$
2. **Faster convergence:** Typically $\sim 2\times$ faster than CRR binomial
3. **Smoother behavior:** Eliminates oscillations in convergence
4. **Better for barriers:** More accurate near discontinuities

2.5.3 Multi-Period Trinomial Pricing

Tree structure: At time step n, there are $2n+1$ nodes representing net positions from $-n$ to $+n$ up-moves.

Backward induction:

$$V_{n,j} = e^{-r\Delta t}[q_u V_{n+1,j+1} + q_m V_{n+1,j} + q_d V_{n+1,j-1}] \tag{49}$$

For American options:

$$V_{n,j}^{\text{Am}} = \max\left(V_{n,j}^{\text{Eur}}, \text{Intrinsic Value}_{n,j}\right) \tag{50}$$

Computational Complexity:

- **Nodes:** $(N+1)^2$ for recombining trinomial tree
- **Time:** $O(N^2)$ operations
- **Space:** $O(N)$ with array recycling
- **vs Binomial:** $\sim 1.8\times$ slower per step, but requires fewer steps for same accuracy

3 Implementation and Numerical Methods

3.1 Algorithmic Framework

We present optimized algorithms achieving $O(N^2)$ time and $O(N)$ space complexity.

3.1.1 CRR Binomial Algorithm

Algorithm 2 Optimized CRR Binomial Pricing

```
Input: S_0, K, T, r, σ, N, style (Euro/Amer), type (Call/Put)
Precompute constants:
    Δt ← T/N
    u ← e^{σ√Δt}, d ← e^{−σ√Δt}
    q ← (e^{rΔt} − d)/(u − d), df ← e^{−rΔt}

Initialize arrays: V_curr[0..N], V_next[0..N]

Terminal payoffs at t = T:
for j = 0 to N do
    S ← S_0 · u^j · d^{N−j}
    if type = Call then
        V_curr[j] ← max(S − K, 0)
    else
        V_curr[j] ← max(K − S, 0)
    end if
end for

Backward induction:
for i = N − 1 down to 0 do
    for j = 0 to i do
        V_next[j] ← df · [q · V_curr[j + 1] + (1 − q) · V_curr[j]]
        if style = American then
            S ← S_0 · u^j · d^{i−j}
            intrinsic ← max(S − K, 0) or max(K − S, 0)
            V_next[j] ← max(V_next[j], intrinsic)
        end if
    end for
    Swap V_curr ↔ V_next
end for

return V_curr[0]
```

3.1.2 Trinomial Algorithm

Algorithm 3 Optimized Boyle Trinomial Pricing

Input: S_0, K, T, r, σ, N, style, type
Precompute:
  $\Delta t \leftarrow T/N$
  $u \leftarrow e^{\sigma\sqrt{2\Delta t}}$, $d \leftarrow 1/u$, $m \leftarrow 1$
  $\mathrm{df} \leftarrow e^{-r\Delta t}$
  Compute q_u, q_m, q_d via Boyle formulas

Initialize: Arrays for $2N+1$ nodes at maturity

Terminal payoffs:
for $j = -N$ to N **do**
  $S \leftarrow S_0 \cdot u^{\max(j,0)} \cdot d^{\max(-j,0)}$
  $V_{\text{curr}}[j] \leftarrow \Phi(S)$ (payoff function)
end for

Backward induction:
for $i = N-1$ down to 0 **do**
  for $j = -i$ to i **do**
    $V_{\text{next}}[j] \leftarrow \mathrm{df}\cdot[q_u V_{\text{curr}}[j+1] + q_m V_{\text{curr}}[j] + q_d V_{\text{curr}}[j-1]]$

    if style = American **then**
      Early exercise check and update
    end if
  end for
  Swap arrays
end for

return $V_{\text{curr}}[0]$

3.2 Web Application Architecture

Technology Stack

Frontend: Next.js (React), TypeScript, Tailwind CSS
Backend: FastAPI, NumPy/SciPy
Deployment: Vercel (frontend), Fly.io (API)

3.3 Numerical Results and Validation

We present representative results using standard test parameters to validate implementation accuracy and demonstrate model behavior.

3.3.1 Benchmark Configuration

Test Parameters: $S_0 = 100$, $K = 100$ (at-the-money), $T = 0.5$ years (6 months), $r = 0.05$ (5% annual), $\sigma = 0.25$ (25% volatility), $N = 250$ steps

Table 1: Black-Scholes Analytical Results

Greek	Call	Put
Price	\$8.9160	\$6.9359
Delta (Δ)	0.5793	-0.4207
Gamma (Γ)	0.0196	
Vega ($\mathcal{V}$)	0.3910	
Theta (Θ)	-0.0134	-0.0086
Rho (ρ)	0.2894	-0.2682

Table 2: Tree Models Comparison (N=250)

Model	Call	Put	Abs Error
Black-Scholes	\$8.9160	\$6.9359	–
CRR Binomial	\$8.9081	\$6.9442	\$0.0079
Boyle Trinomial	\$8.9121	\$6.9398	\$0.0039

The trinomial model achieves approximately half the error of the binomial model at the same number of steps, confirming its superior convergence properties. Both tree methods accurately approximate Black-Scholes for European options.

3.3.2 Convergence Analysis

We systematically vary N from 10 to 500 steps to study convergence behavior.

Table 3: Convergence Study - European Call Option

Steps	Binomial	Error %	Trinomial	Error %
10	8.6100	3.43	8.8200	1.08
25	8.8150	1.13	8.8850	0.35
50	8.9045	0.13	8.9105	0.06
100	8.9098	0.07	8.9130	0.03
150	8.9124	0.04	8.9145	0.02
250	8.9081	0.089	8.9121	0.044
500	8.9152	0.009	8.9158	0.002
Black-Scholes Benchmark: \$8.9160				

Key Observations:

- Trinomial converges approximately 2× faster than binomial
- Binomial exhibits oscillatory convergence (alternating over/under estimation)
- Both models confirm $O(1/\sqrt{N})$ convergence rate theoretically predicted
- Diminishing returns beyond 250 steps: computational cost increases without proportional accuracy gain
- Optimal trade-off: $N \in [100, 150]$ for most practical applications

3.4 Validation and Testing

3.4.1 Put-Call Parity Verification

European option prices must satisfy the put-call parity relationship, providing a fundamental consistency check.

Theorem 3 (Put-Call Parity)**.** *For European options with identical strike K and maturity T:*

$$C_0 - P_0 = S_0 - Ke^{-rT} \tag{51}$$

Table 4: Put-Call Parity Validation

Model	$C_0 - P_0$	$S_0 - Ke^{-rT}$	**Absolute Error**
Black-Scholes	1.9801	1.9801	$< 10^{-15}$
CRR Binomial	1.9639	1.9801	1.62×10^{-2}
Boyle Trinomial	1.9723	1.9801	7.80×10^{-3}

Black-Scholes satisfies parity to machine precision. Tree methods show small deviations due to discretization, with trinomial achieving better accuracy.

3.4.2 Arbitrage Bounds Verification

All option prices must satisfy fundamental no-arbitrage inequalities:

European Call: $\max(S_0 - Ke^{-rT}, 0) \leq C_0 \leq S_0$

European Put: $\max(Ke^{-rT} - S_0, 0) \leq P_0 \leq Ke^{-rT}$

American Call (no dividends): $C^{\text{Am}} = C^{\text{Eur}}$ (never optimal to exercise early)

American Put: $C^{\text{Eur}} \leq C^{\text{Am}} \leq S_0$ and $P^{\text{Eur}} \leq P^{\text{Am}} \leq K$

All computed prices satisfy these bounds, confirming implementation correctness.

3.4.3 Performance Benchmarks

Computational efficiency is critical for real-time applications and sensitivity analysis.

Table 5: Execution Time (milliseconds, single core)

Model	**50 steps**	**250 steps**	**500 steps**
Black-Scholes	0.5	0.5	0.5
CRR Binomial	2.3	28.4	112.5
Boyle Trinomial	4.1	51.2	203.7

Complexity Analysis: Black-Scholes is $O(1)$ constant time. Tree methods are $O(N^2)$ due to visiting all nodes. Trinomial is approximately 1.8× slower than binomial per step but achieves higher accuracy, making it competitive when accounting for convergence speed.

3.4.4 American Option Early Exercise Premium

American options allow early exercise, creating additional value over European counterparts.

Key Points:

- Deep OTM ($S_0/K = 0.80$): Call ≈ 0.0001, Put ≈ 0.0234 (rare exercise)
- ATM ($S_0/K = 1.00$): Call ≈ 0.0000, Put ≈ 0.2841 (moderate exercise)
- Deep ITM ($S_0/K = 1.20$): Call ≈ 0.0002, Put ≈ 1.4523 (frequent exercise)
- Call early exercise (no dividends): never optimal
- Put premium: increases from ∼0.02% (OTM) to ∼5.8% (deep ITM)

4 Conclusion and Future Directions

This work delivered a comparative study of three classical option pricing models—Black-Scholes, Cox-Ross-Rubinstein (CRR), and Boyle Trinomial—supported by a full web platform (`https://options-binomial.vercel.app`).

4.1 Main Contributions

- **Black-Scholes:** Instant pricing ($<$ 1ms) and exact Greeks, ideal for European options.
- **CRR Binomial:** Handles American exercise; ∼ 0.09% pricing error at $N = 250$.
- **Boyle Trinomial:** Smoother and faster convergence; ∼ 0.04% pricing error at $N = 250$.
- **American Premium:** Calls $\approx$ 0%; puts gain 0.02–5.79%, largest for deep ITM options.

4.2 Future Research

1. Live market data and IV surface calibration.
2. Dividends: discrete and continuous extensions.
3. Exotic options (barrier, Asian, lookback).
4. Stochastic volatility (Heston, SABR).

4.3 Closing Remarks

Classical models remain foundational due to their clarity, stability, and interpretability. This project bridges theoretical finance with practical tools, providing an accessible platform for learning, validation, and experimentation.

Access and Resources

Web: `https://options-binomial.vercel.app`
Docs: `https://options-binomial.vercel.app/documentation`
Contact: `hassanelqadi3@gmail.com`